An overview of GPS inter-frequency carrier phase combinations.

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Abstract

A comprehensive study of the inter-frequency combinations available from dual-frequency GPS carrier phase observations is presented. There are three criteria that can be examined: 1) the final combination is a 'widelane' (with a wavelength greater than the L2 frequency); 2) the ionospheric impact is reduced compared to the L1 frequency; and 3) the noise is reduced compared to the L1 frequency. All three criteria must be examined to deduce the three most common combinations: the widelane, the ionosphere-free and the narrowlane. The integer nature of the ionosphere-free ambiguities is confirmed. The amplification of the wavelength of a second combination is discussed, along with the so-called 'odd/even' relationship and its impact on *Teunissen*'s [1995] hypothesis of the incompatibility of certain double combinations.

Introduction

We consider simplified versions of the GPS L1 and L2 carrier phase observations:

$$L1[\mathrm{m}] = \rho + \lambda_1 N_1 - I\,, \qquad (1)$$

$$L2[\mathrm{m}] = \rho + \lambda_2 N_2 - q^2 I\,, \qquad (2)$$

where ρ represents the geometric quantities invariant with frequency (troposphere, clocks, geometric range), $\lambda_{1/2}$ is the wavelength associated with each frequency, $N_{1/2}$ is the associated cycle ambiguity, I is the ionospheric propagation delay on the L1 frequency and q is the ratio of the carrier frequencies f_1/f_2 = 77/60. The random noise and multipath of each measurement are ignored for the moment. The general expression for a combined observation is:

$$LC[\mathrm{m}] = \alpha L1 + \beta L2\,, \qquad (3)$$

which can be expanded explicitly as:

$$LC[\mathrm{m}] = \rho(\alpha+\beta) + \alpha\lambda_1 N_1 + \beta\lambda_2 N_2 - I(\alpha + \beta q^2)\,. \qquad (4)$$

If we wish to impose the constraints that the geometric portion remains unchanged and that the resulting ambiguity is still an integer, then we can equate each term with the generic expression:

$$LC[\mathrm{m}] = \rho + \lambda N - I\eta\,, \qquad (5)$$

to obtain:

$$\alpha + \beta = 1\,, \tag{6a}$$
$$\alpha\lambda_1 N_1 + \beta\lambda_2 N_2 = \lambda N\,, \tag{6b}$$
$$(\alpha + \beta q^2) = \eta\,. \tag{6c}$$

Equation (6b) gives an expression for the combined ambiguity:

$$N = \frac{\alpha\lambda_1 N_1}{\lambda} + \frac{\beta\lambda_2 N_2}{\lambda}\,. \tag{7}$$

For N to be an integer, then the parameters:

$$i = \frac{\alpha\lambda_1}{\lambda}\,, \quad j = \frac{\beta\lambda_2}{\lambda}\,, \tag{8}$$

must also be integers. The easiest way to achieve this is to define them as such and then re-arrange to compute α and β thus:

$$\alpha = \frac{i\lambda}{\lambda_1}\,, \quad \beta = \frac{j\lambda}{\lambda_2}\,. \tag{9}$$

This result indicates that using the α and β parameters to compute a combination with metric units implicitly converts the *L1* and *L2* measurements into cycles before combining them.

The wavelength of the combination can now be deduced from equation (6a):

$$\lambda = \frac{\lambda_1\lambda_2}{i\lambda_2 + j\lambda_1}\,. \tag{10}$$

Substituting the generic relationship $\lambda = c/f$, where c is the vacuum speed of light, gives the frequency of the combination:

$$f = if_1 + jf_2\,. \tag{11}$$

Given that we are usually concerned with the integer nature of the ambiguities, it makes sense to write our observation equations explicitly in units of cycles:

$$\frac{L1}{\lambda_1} = \Phi_1[\mathrm{cy}] = \frac{\rho}{\lambda_1} + N_1 - \frac{I}{\lambda_1}\,, \tag{12}$$
$$\frac{L2}{\lambda_2} = \Phi_2[\mathrm{cy}] = \frac{\rho}{\lambda_2} + N_2 - \frac{q^2 I}{\lambda_2}\,, \tag{13}$$

and substituting equation (9) into equation (4),

$$
\begin{aligned}
\frac{LC}{\lambda} = \Phi_{i,j}[\text{cy}] &= i\Phi_1 + j\Phi_2 \\
&= \rho\left(\frac{i}{\lambda_1} + \frac{j}{\lambda_2}\right) + iN_1 + jN_2 - I\left(\frac{i}{\lambda_1} + \frac{j}{\lambda_2} q^2\right) \\
&= \frac{\rho}{\lambda} + iN_1 + jN_2 - \frac{I}{\lambda_1}(i + jq)\,. \qquad (14)
\end{aligned}
$$

Using the law of error propagation, the noise of a combination can be expressed as:

$$
\begin{aligned}
\acute{o}_{\ddot{O}_{i,j}}[\text{cy}] &= \sqrt{i^2 \acute{o}^2_{\ddot{O}_1} + j^2 \acute{o}^2_{\ddot{O}_2}} \\
&= \sqrt{(i^2 + \kappa^2 j^2)\sigma^2_{\Phi_1}}\ ,\ \text{if}\ \sigma_{\Phi_2} = \kappa\sigma_{\Phi_1}\,, \qquad (15)
\end{aligned}
$$

$$
\begin{aligned}
\sigma_{LC}[\text{m}] &= \sqrt{\alpha^2\sigma^2_{L_1} + \beta^2\sigma^2_{L_2}} \\
&= \sqrt{(\alpha^2 + \upsilon^2\beta^2)\sigma^2_{L_1}}\ ,\ \text{if}\ \sigma_{L_2} = \upsilon\sigma_{L_1}\,. \qquad (16)
\end{aligned}
$$

For a biased error:

$$
\begin{aligned}
\ddot{a}_{\ddot{O}_{i,j}}[\text{cy}] &= i\ddot{a}_{\ddot{O}_1} + j\ddot{a}_{\ddot{O}_2} \\
&= (i + \gamma j)\delta_{\Phi_1}\,,\ \text{if}\ \delta_{\Phi_2} = \gamma\delta_{\Phi_1}\,, \qquad (17)
\end{aligned}
$$

$$
\begin{aligned}
\delta_{LC}[\text{m}] &= \alpha\delta_{L_1} + \beta\delta_{L_2} \\
&= (\alpha + \psi\beta)\delta_{L_1}\,,\ \text{if}\ \delta_{L_2} = \psi\delta_{L_1}\,. \qquad (18)
\end{aligned}
$$

The Widelane Criterion

We turn now to choosing meaningful values of *i* and *j* to obtain useful combinations. It is generally recognised that the so-called widelane combinations are useful when trying to resolve ambiguities. We shall show that there are only a finite number of widelane combinations. Our derivation is similar to that of *Cocard and Geiger* [1992].

We start by looking at the denominator of equation (10). We can set up an inequality that maximises the value of λ:

$$
\forall\ \lambda_1 > i\lambda_2 + j\lambda_1 > 0\ \exists\ \lambda > \lambda_2\,, \qquad (19)
$$

i.e. we define a 'widelane' as being an observation combination with a resulting wavelength greater than the L2 wavelength. Re-arranging this inequality and using the identity $q = \lambda_2/\lambda_1$ we obtain:

$$1 - qi > j > -qi . \quad (20)$$

The range of this inequality is 1, so unless *qi* is an integer, something we will discuss in a moment, there can only ever be one value of *j* for any *i*. Hence:

$$j = \lceil (-qi) \rceil , \quad (21)$$

where $\lceil \cdot \rceil$ is the integer ceiling function that rounds its argument upward to $+\infty$ (i.e. $\lceil -1.5 \rceil = -1$).

Substituting to compute the wavelength directly:

$$\lambda(i) = \frac{\lambda_2}{qi + \lceil (-qi) \rceil} . \quad (22)$$

This equation allows us to investigate the cyclic relationship that derives from the fact that *q* is not an irrational number, but the quotient of two integers, 77/60. If we define the period *P* so that $\lambda(i) = \lambda(i + P)$, then we have to solve:

$$qi + \lceil (-qi) \rceil = q(i + P) + \lceil (-q(i + P)) \rceil .$$

This equation will only be true when *qP* is an integer, in other words, $P = 60$. This suggests that the range of *i* is [1,60], however we must consider one more point. The concept of error propagation implies that the noise of the observation combination increases with increasing values of *i*. It makes sense therefore, to minimise the absolute values of *i*, such that $i \in \Im[-29,30] \neq 0$. Figure 1 plots the wavelengths of all the possible combinations chosen in this way. Unfortunately not all are linearly independent from one another — some values of *i* and *j* can be divided, without remainder, by a common integer. The final list of independent widelanes is given in Table 1.

Table 2 summarises all the possible widelane combinations for which the index *i* lies in the range ±10. The wavelength for each combination is given, along with its ratio to the *L1* wavelength. This indicates by how much the geometric portion of the *L1* observation is reduced in the combined observation. The amplification of the noise, ionosphere and multipath are given in units of cycles and length. The noise is propagated as a random error; the ionosphere and multipath are propagated as biases. The noise of *L2* is not considered to be equal to *L1*, as is often assumed. The κ factor in equation (15) was computed from some dual-frequency data collected with a Trimble geodetic receiver over a short baseline. As such, it contains a certain multipath component, but this is acceptable for our purposes. The ionospheric factor in cycles is computed using equation (17) with $\gamma = q$ (c.f. equation (14)). The equivalent factor in length units (υ) is simply η in equation (6c). The multipath is considered to exhibit the maximum bias possible, i.e. a full quarter-cycle simultaneously on both *L1* and *L2* [*Georgiadou and Kleusberg*, 1988]. To compute the in-phase maximum, the absolute values of indices *i* and *j* are used in equation (17) with $\gamma = 1$. For the multipath in length units, $\psi = \mathrm{q}$. These values can therefore be considered as an upper limit.

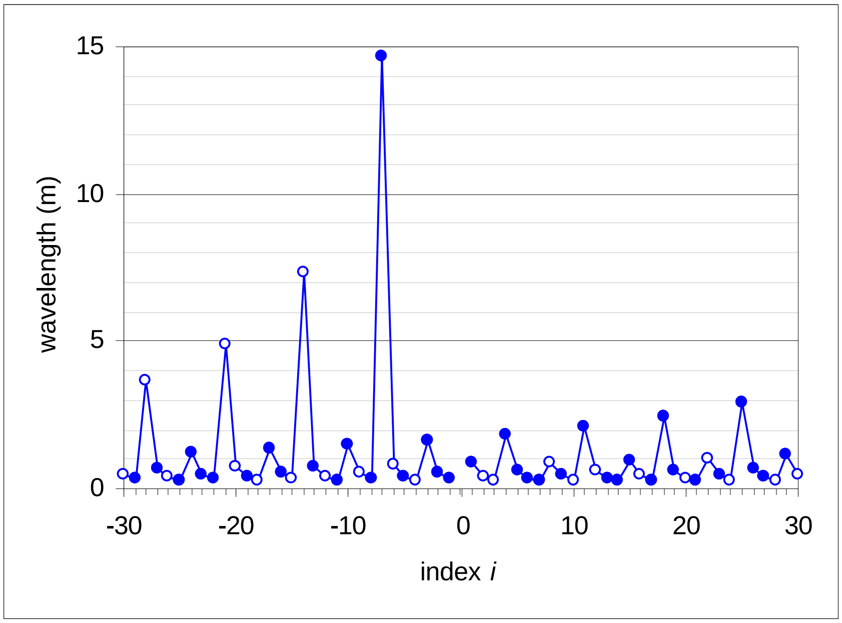

Figure 1. GPS carrier phase combinations with widelane wavelengths. Combinations represented with open circles are not independent. (After Cocard and Geiger, [1992])

Table 1. Independent widelane combinations. (After Cocard and Geiger [1992] who were missing (4,-5).)

i	*j*	λ(cm)	*i*	*j*	λ(cm)	*i*	*j*	λ(cm)
–29	38	31.2	–8	11	33.3	13	–16	35.7
–27	35	69.8	–7	9	1465.3	14	–17	25.3
–25	33	26.6	–5	7	41.9	15	–19	97.7
–24	31	122.1	–3	4	162.8	17	–21	29.9
–23	30	50.5	–2	3	56.4	18	–23	244.2
–22	29	31.9	–1	2	34.1	19	–24	63.7
–19	25	39.6	1	–1	86.2	21	–26	25.7
–17	22	133.2	4	–5	183.2	23	–29	47.3
–16	21	52.3	5	–6	58.6	25	–32	293.1
–13	17	77.1	6	–7	34.9	26	–33	66.6
–11	15	27.6	7	–8	24.8	27	–34	37.6
–10	13	146.5	9	–11	44.4	29	–37	112.7
			11	–14	209.3			

Table 2. Widelane combination characteristics.

LC	i	j	α	β	λ_{LC}	λ_1/λ_{LC}	Amplification (cycles)			Amplification (length)		
							Noise	ion	$\lvert mp \rvert$	noise	ion	$\lvert mp \rvert$
L1	1	0	1	0	19.0	1	1	1	0.25	1	1	1
L2	0	1	0	1	24.4	0.78	1.17	1.28	0.25	1.5	1.6	1.28
WL	1	-1	4.5294	-3.5294	86.2	0.22	1.54	-0.28	0.50	7.0	-1.3	9.06
W1	-1	2	-1.7907	2.7907	34.1	0.56	2.54	1.57	0.75	4.6	2.8	5.37
W2	-2	3	-5.9231	6.9231	56.4	0.34	4.04	1.85	1.25	12.0	5.5	14.81
W3	-3	4	-25.6667	26.6667	162.8	0.12	5.56	2.13	1.75	47.5	18.3	59.89
W4	4	-5	38.5	-37.5	183.2	0.10	7.08	-2.42	2.25	68.2	-23.3	86.63
	5	-6	15.4	-14.4	58.6	0.32	8.61	-2.70	2.75	26.5	-8.3	33.88
	-5	7	-11	12	41.9	0.45	9.59	3.98	3.00	21.1	8.8	26.40
	6	-7	11	-10	34.9	0.55	10.15	-2.98	3.25	18.6	-5.5	23.83
	7	-8	9.1356	-8.1356	24.8	0.77	11.68	-3.27	3.75	15.2	-4.3	19.58
EW	-7	9	-539	540	1465.3	0.01	12.64	4.55	4.00	972.9	350.4	1232.0
	-8	11	-14	15	33.3	0.57	15.14	6.12	4.75	26.5	10.7	33.25
	9	-11	21	-20	44.4	0.43	15.69	-5.12	5.00	36.6	-11.9	46.67
	-10	13	-77	78	146.5	0.13	18.19	6.68	5.75	140.1	51.5	177.10

Those combinations that have appeared in the literature previously are labelled in Table 2 with a two-character code in column 1. Of these labels, only '*WL*' is commonly used (and not universally, the Berne group uses *L5*, see e.g., *Beutler et al*. [1990]). Of the other combinations, the extra-widelane combination (*EW*) has received some recent attention [*Han and Rizos*, 1995; *Han*, 1995; *Han*, 1997], as have *W1*–*W4* to a lesser extent [e.g., *Han and Rizos*, 1996].

Other Criteria (1)

Considering again the denominator in equation (10) we can see that:

$$\forall \;\; i\lambda_2 + j\lambda_1 \ge \lambda_1 \;\; \exists \;\; \lambda \le \lambda_2, \qquad (23)$$

i.e., there are an infinite number of narrowlane combinations. To determine further useful combinations, we must therefore examine some other criteria that the combined observation can fulfil. The most straightforward criterion is to reduce the ionospheric path delay. By considering our generalised observation equation in units of cycles (equation (14)), we can see that to reduce the amount of ionosphere over that experienced on the *L1* frequency, the following inequality must hold.

$$\left| i + jq \right| < 1,$$

or in terms of j:

$$\frac{-1-i}{q} < j < \frac{1-i}{q}, \qquad (24)$$

which is in a form similar to equation (20). The range of this inequality is ~1.56, indicating that there can be two integer values of *j* for some values of *i*. We should note that equation (24) is not limited to narrowlane combinations. To compute values of *j*:

$$j^{A} = \left\lceil \frac{-1-i}{q} \right\rceil, \qquad (25)$$

$$j^{B} = \left\lfloor \frac{1-i}{q} \right\rfloor, \text{ ignore if } j^{B} = j^{A}. \qquad (26)$$

where $\lfloor \cdot \rfloor$ is the integer floor function that rounds its argument downward to $-\infty$ (i.e. $\lfloor -1.5 \rfloor = -2$).

Substituting these functions of j into equation (10) gives:

$$\lambda(i\{j^{A}\}) = \frac{\lambda_2}{qi + \left\lceil \frac{-1-i}{q} \right\rceil}, \quad \lambda(i\{j^{B}\}) = \frac{\lambda_2}{qi + \left\lfloor \frac{1-i}{q} \right\rfloor}. \qquad (27)$$

Unlike the widelane combinations, there is no cyclic variation, because there are an unlimited number of choices for the index *i*. Even discarding common multiples, there will always be a prime number to use for *i*. However, the combination $i = 77$, $j^{A} = j^{B} = -60$, sets the ionospheric factor to zero, which suggests that there is no more information to be gained from going beyond this point. At the same time, negative values of *i* give only negative wavelengths. Therefore, we will only consider $i \in \Im[1,77]$.

Figure 2 plots the values of the ionosphere factor ($i + jq$) against the chosen range of *i*. The upper portion of the plot shows the values of j^{B}, the lower portion j^{A}. As in Figure 1, the combinations that are common multiples have been represented with open symbols. Table 3 reveals that all but one of these combinations is a narrowlane combination except for $\lambda(i = 1)$, which is the *WL* widelane combination (1,–1). This result helps to clarify the importance of this particular combination. Of all the possible combinations from dual-frequency GPS carrier phase measurements, it is the only one that is both a widelane and reduces the impact of the ionosphere.

A summary of some of the reduced-ionosphere combinations is given in Table 4. The entries for this table were computed in the same way as for the widelane summary (Table 2). Only those combinations for which $i \in \Im[2,10]$ are shown, along with the (77,–60) combination.

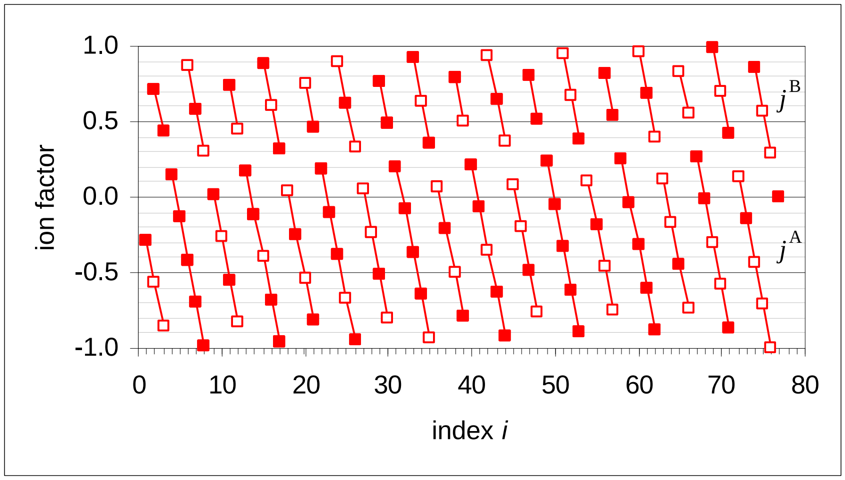

Figure 2. GPS carrier phase combinations with ionospheric factors less than 1. Combinations represented with open symbols are not independent.

Table 3. Independent reduced-ionosphere combinations.

i	*j*	λ(cm)	*i*	*j*	λ(cm)	*i*	*j*	λ(cm)
1	–1	86.2	25	–19	1.9	52	–41	0.9
2	–1	15.6	26	–21	2.0	53	–41	0.9
3	–2	13.2	29	–22	1.6	53	–42	0.9
4	–3	11.4	29	–23	1.7	55	–43	0.9
5	–4	10.1	30	–23	1.6	56	–43	0.8
6	–5	9.0	31	–24	1.5	57	–44	0.8
7	–5	6.1	32	–25	1.5	58	–45	0.8
7	–6	8.2	33	–25	1.4	59	–46	0.8
8	–7	7.5	33	–26	1.5	60	–47	0.8
9	–7	5.4	34	–27	1.5	61	–47	0.8
11	–8	4.0	35	–27	1.4	61	–48	0.8
11	–9	4.8	37	–29	1.3	62	–49	0.8
13	–10	3.7	38	–29	1.2	65	–51	0.8
14	–11	3.5	39	–31	1.3	67	–52	0.7
15	–11	3.0	40	–31	1.2	68	–53	0.7
16	–13	3.2	41	–32	1.2	69	–53	0.7
17	–13	2.8	43	–33	1.1	71	–55	0.7
17	–14	3.1	43	–34	1.2	71	–56	0.7
19	–15	2.6	44	–35	1.1	73	–57	0.7
21	–16	2.2	47	–36	1.0	74	–57	0.6
21	–17	2.5	47	–37	1.0	75	–58	0.6
22	–17	2.2	48	–37	1.0	75	–59	0.7
23	–18	2.1	49	–38	1.0	76	–59	0.6
24	–19	2.1	50	–39	1.0	77	–60	0.6
			51	–40	1.0			

Table 4. Reduced-ionosphere combination characteristics.

LC	*i*	*j*	α	β	λ_{LC}	λ_1/λ_{LC}	Amplification (cycles)			Amplification (length)		
							noise	ion	|mp|	noise	ion	|mp|
L1	1	0	1	0	19.0	1	1	1	0.25	1	1	1
L2	0	1	0	1	24.4	0.78	1.17	1.28	0.25	1.5	1.6	1.28
	2	-1	1.6383	-0.6383	15.6	1.22	2.32	0.72	0.75	1.9	0.6	2.46
	3	-2	2.0811	-1.0811	13.2	1.44	3.80	0.43	1.25	2.6	0.3	3.47
N1	4	-3	2.4063	-1.4063	11.4	1.66	5.32	0.15	1.75	3.2	0.1	4.21
N2	5	-4	2.6552	-1.6552	10.1	1.88	6.85	-0.13	2.25	3.6	-0.1	4.78
	6	-5	2.8519	-1.8519	9.0	2.10	8.38	-0.42	2.75	4.0	-0.2	5.23
	7	-5	2.2552	-1.2552	6.1	3.10	9.12	0.58	3.00	2.9	0.2	3.87
	7	-6	3.0112	-2.0112	8.2	2.32	9.91	-0.70	3.25	4.3	-0.3	5.59
	8	-7	3.1429	-2.1429	7.5	2.55	11.44	-0.98	3.75	4.5	-0.4	5.89
EN	9	-7	2.5385	-1.5385	5.4	3.55	12.16	0.017	4.00	3.4	0.005	4.51
IF	77	-60	2.5457	-1.5457	0.6	30.25	104.15	0	34.25	3.4	0	4.53

The combinations in Table 4 denoted as '*N1*' and '*N2*' have been investigated previously due to their combination of ionosphere-reduction properties and reasonable wavelengths [*Wübbena*, 1989; *Wanninger*, 1991]. Of all the reduced-ionosphere combinations, these two are the best in this regard, and most of the others are included for the sake of completeness and curiosity. In the latter category is the combination denoted '*EN*', for extra-narrowlane. As far as this author knows, this combination has not been discussed previously in the open literature. This is unusual, because it almost completely removes the ionospheric impact and yet has a wavelength almost 10 times longer than the 'completely' ionosphere-free combination (77,–60) denoted here as '*IF*'. The noise and multipath factors are almost the same as the *IF* combination (compare their respective α and β values), yet the longer wavelength might allow for the ambiguities to be solved directly under some circumstances. Depending on the amount of ionospheric delay experienced, this would give a solution with almost identical statistical results to an *IF* solution, but precluding the initial step of resolving the widelane ambiguities (more about that shortly).

At this point it is particularly instructive to consider the *IF* combination (77,–60) more closely. It is often stated in the literature that the ambiguity parameter in this combination is not an integer value. It should be obvious from our derivation that it is. The problem with this combination is that the effective wavelength is drastically reduced compared to *L1* and *L2*. This greatly amplifies the observation noise in terms of cycles and makes direct resolution of the *IF* ambiguities almost impossible.

Other Criteria (2)

Finally, we must realise that the last commonly referred to combination has not so far turned up in our investigation. This is the combination (1,1), often referred to simply as the ‘narrowlane’. It has the unusual property of reducing the noise (in terms of length units). There are obviously no values of *i* and *j* that satisfy the inequality:

$$\sqrt{i^2 + j^2} < 1, \qquad (28)$$

however, the inequality:

$$\sqrt{\acute{a}^2 + \hat{a}^2} < 1, \qquad (29)$$

is another matter. Note that for the moment we consider the noise of *L2* to be equal to that of *L1*. The minimum value of equation (29) is $\sqrt{½} \approx 0.71$ ($\alpha = \beta = 0.5$), i.e. we cannot expect a great reduction in the noise and it must be at the expense of some other quantity. In this case there is a reduction in the wavelength over *L1*, i.e. all the solutions to equation (29) must be narrowlanes. It turns out that the solutions to equation (29) exist only for values of *i* and $j \geq 1$. Only those unique combinations with a wavelength ≥ 5cm are summarised in Table 5.

Table 5. Noise-reduction combination characteristics.

LC	*i*	*j*	α	β	λ_{LC}	λ_1/λ_{LC}	Amplification (cycles)			Amplification (length)		
							noise	ion	|mp|	noise	ion	|mp|
L1	1	0	1	0	19.0	1	1	1	0.25	1	1	1
L2	0	1	0	1	24.4	0.78	1.17	1.28	0.25	1.5 (1)	1.6	1.28
NL	1	1	0.5620	0.4380	10.7	1.78	1.54	2.28	0.50	0.86 (0.71)	1.3	1.12
	1	2	0.3909	0.6091	7.4	2.56	2.54	3.57	0.75	0.99 (0.72)	1.4	1.17
N3	2	1	0.7196	0.2804	6.8	2.78	2.32	3.28	0.75	0.83 (0.77)	1.2	1.08
	1	3	0.2996	0.7004	5.7	3.34	3.65	4.85	1.00	1.09 (0.76)	1.5	1.20
N4	3	1	0.7938	0.2062	5.0	3.78	3.22	4.28	1.00	0.85 (0.82)	1.1	1.06

The entries for Table 5 were computed in the same manner as the previous two summaries. The only exception is for the noise factors in length units. The values in brackets are the results from setting the *L2* observation noise equal to that of *L1*. We have chosen to highlight combinations ‘*N3*’ and ‘*N4*’ only because they minimise the contribution of the *L2* noise and the combined multipath. The generic narrowlane combination $i = 1, j = 1$ is denoted ‘*NL*’.

<u>Resolving Ambiguities</u>

The ionosphere-free nature of the *IF* combination is obviously a very desirable property, unfortunately the very short wavelength and comparatively large noise generally means that the numerical value of the ambiguities cannot be estimated directly. It is possible to fix the ambiguities indirectly however, by first fixing the ambiguities of another combination. If we considering the general case of two combinations with integer ambiguities $N_{i,j}$ and $N_{k,l}$, where:

$$N_{i,j} = iN_1 + jN_2 \,, \qquad (30)$$

and $$N_{k,l} = kN_1 + lN_2 \,, \qquad (31)$$

we can rewrite the first combination in terms of N_2 and substitute the result into the second combination to obtain:

$$N_{k,l} = (k - i\frac{l}{j})N_1 + \frac{l}{j}N_{i,j} \,. \qquad (32)$$

In terms of the widelane ($N_{i,j} = N_{1,-1}$) and ionosphere-free combinations ($N_{k,l} = N_{77,-60}$):

$$N_{77,-60} = 17N_1 + 60N_{1,-1} \,. \qquad (33)$$

Multiplying both sides of this equation by $\lambda_{77,-60}$ makes it equivalent to equation (6b). In other words, resolving the widelane ambiguities and substituting them into the *IF* combination reduces the *IF* ambiguities to the *L1* ambiguities while amplifying the wavelength by a factor of 17 (to 10.7cm). This can then often allow the *L1* ambiguities to be solved for directly. As a downside, these *L1* ambiguities are very sensitive to incorrect widelane ambiguities ($60\lambda_{77,-60}$ = 37.8cm). However, as a check on the original widelane determination, the *L1* ambiguities can be substituted back into the ionosphere-free combination to amplify the wavelength by a factor of 60 and isolate the *L2* ambiguities:

$$N_{77,-60} = 77N_1 - 60N_2 \,;\; N_1 = 0 \Rightarrow N_{77,-60} = 60N_2 \,. \qquad (34)$$

The amplification factors and effective wavelengths available after resolving the widelane ambiguities are summarised in Table 6. Table 7 summarises the combinations affected by resolving the narrowlane (*NL*) ambiguities. The amplified widelane in this case (effective wavelength = 172.4cm), is the "extra wide lane" referred to by *Wübbena* [1988,1989]. The wavelength for the ionosphere-free combination is amplified even further by resolving the narrowlane combination. However, it is hard to imagine when this might be effective, due to either the necessity of overcoming the ionosphere when resolving the narrowlane ambiguities in the first place, or the redundancy of the option over short baselines where the effect should largely cancel. Nevertheless, should observable noise be a larger consideration than the ionosphere, the use of the widelane and narrowlane combinations (as the least noisy of all the possible combinations) should prove useful.

Table 6. Amplification factors and effective wavelengths of combinations affected by resolving the widelane (WL) ambiguities.

LC	*EW*	*NL*	*N3*	*N4*	*EN*	*IF*
factor	2	2	3	4	2	17
λ(cm)	2930.5	21.4	20.5	20.1	10.7	10.7

Table 7. Amplification factors and effective wavelengths of combinations affected by resolving the narrowlane (NL) ambiguities.

LC	WL	W1	W2	W3	W4	EW	N1	N2	N4	EN	IF
factor	2	-3	-5	-7	9	-16	7	9	2	16	137
λ(cm)	172.4	-102.2	-281.8	-1139.6	1648.4	-23444.2	80.1	90.9	10.1	85.9	86.2

Admissible Transformations to L1 and L2

Teunissen [1995] has implied that there are only a limited number of admissible transformations that unambiguously determine the L1 and L2 ambiguities from two linear combinations. One apparent consequence is that the widelane and narrowlane can not be paired together. This hypothesis is based on the following transformation:

$$\begin{bmatrix} N_{i,j} \\ N_{k,l} \end{bmatrix} = \begin{bmatrix} i & j \\ k & l \end{bmatrix} \begin{bmatrix} N_1 \\ N_2 \end{bmatrix}, \qquad (35)$$

which is merely equations (30) and (31) in matrix form. To guarantee a one-to-one reverse transformation, that is that N_1 and N_2 must always be integers given that $N_{i,j}$ and $N_{k,l}$ are integers, the inverse of the transformation matrix must have only integer components, i.e. in:

$$\begin{bmatrix} N_1 \\ N_2 \end{bmatrix} = \frac{1}{il - jk} \begin{bmatrix} l & -j \\ -k & i \end{bmatrix} \begin{bmatrix} N_{i,j} \\ N_{k,l} \end{bmatrix}, \qquad (36)$$

the determinant $il - jk$ must equal ±1. For the widelane and narrowlane combinations the reverse transformation is:

$$\begin{bmatrix} N_1 \\ N_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} N_{1,1} \\ N_{1,-1} \end{bmatrix}. \qquad (37)$$

An example is given in *Teunissen* [1995] of $N_{1,1} = 1$ and $N_{1,-1} = 0$, for which non-integer values of N_1 and N_2 result. Unfortunately, there is a slight fallacy here. What this result means is that either $N_{1,1}$ or $N_{1,-1}$ have been incorrectly determined, not that the transformation is inadmissible *per se*. This fact is obvious from the odd/even law which states [*Wübbena*, 1988]:

if $N_{1,-1}$ is even, $N_{1,1}$ has to be even, and

if $N_{1,-1}$ is odd, $N_{1,1}$ has to be odd.

Hence, the above transformation should not be attempted before trying to determine which value of $N_{1,1}$ or $N_{1,-1}$ is incorrect. For example, if the float value of $N_{1,-1}$ in the previous example was 0.4, the value of 1 is more likely to be the true integer value. Table 8 indicates those combinations that combine to form “admissible” transformations according to *Teunissen* [1995]. The odd/even law governs none of these double combinations, and the <u>guarantee</u> of achieving a one-to-one transformation to N_1 and N_2 is of little use if either one or both of the combination ambiguities is incorrectly determined. While ambiguity fixing is an inexact science at best, this latter point indicates that the double combinations highlighted with a ‘**Y**’ in Table 8 should probably be avoided.

The applicability of the odd/even law is also indicated by the determinant of the transformation matrix. In general, the odd or even pattern of the N_1 and N_2 values must be preserved or equally translated to the combinations. In effect, either the N_1 or N_2 factors (*i* and *k or j* and *l*) can be even, or either *all* the factors must be odd. It is not possible for all of them to be even because that would imply non-independent combinations. The result is that the determinant $il - jk$ must be even. Table 8 highlights those double combinations examined in this paper that conform to the odd/even law. Of course, all these combinations and the remaining combinations are constrained by the fact that non-integer results for N_1 and/or N_2 indicate incorrect determination of N_{ij} and/or N_{kl}.

*Table 8. Admissible double combinations according to Teunissen [1995] (**Y**) and double combinations that comply with the odd/even law (*).*

LC			*WL*	*W1*	*W2*	*W3*	*W4*	*EW*	*NL*	*N1*	*N2*	*N3*	*N4*	*EN*	*IF*
			1	-1	-2	-3	4	-7	1	4	5	2	3	9	77
			-1	2	3	4	-5	9	1	-3	-4	1	1	-7	-60
WL	1	-1		**Y**	**Y**	**Y**	**Y**	*	*	**Y**	**Y**		*	*	
W1	-1	2	**Y**		**Y**	*					*				*
W2	-2	3	**Y**	**Y**		**Y**	*			*		*			
W3	-3	4	**Y**	*	**Y**		**Y**	**Y**			*				*
W4	4	-5	**Y**		*	**Y**		**Y**		*		*			
EW	-7	9	*			**Y**	**Y**		*				*	*	
NL	1	1	*					*				**Y**	*	*	
N1	4	-3	**Y**		*		*				**Y**	*		**Y**	
N2	5	-4	**Y**	*		*				**Y**				**Y**	*
N3	2	1			*		*		**Y**	*			**Y**		
N4	3	1	*					*	*			**Y**		*	
EN	9	-7	*					*	*	**Y**	**Y**		*		**Y**
IF	77	-60		*		*					*			**Y**	

Conclusions

This paper has attempted to provide a comprehensive overview of the possible combinations available from dual-frequency GPS carrier-phase observations. All of the combinations have been constrained to integer ambiguities. The integer nature of the ionosphere-free ambiguities has been confirmed. Because all the combinations are implicitly derived from the separate L1 and L2 observations, there is always an admissible reverse transformation of a pair of combinations back to L1 and L2. The failure of such a transformation (i.e. non-integer ambiguities) indicates an incorrect determination of the combination ambiguities. However, several pairs of combinations will always give integer values due to the determinant of the transformation matrix being unity. This suggests that their use should be avoided.

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